

# Dimensional Analysis and Physical Laws

Shinji Tanimoto

(tanimoto@cc.kochi-wu.ac.jp)

Department of Mathematics  
Kochi Joshi University  
Kochi 780-8515, Japan.

## Abstract

Dimensional analysis provides many simple and useful tools for various situations in science. The objective of this paper is to investigate its relations to functions, i.e., the dimensions for functions that yield physical quantities and those for the arguments of functions. In particular, it is shown that there are three types of functions from a viewpoint of dimensional analysis. The results presented here seem to be useful in several aspects and can also be applied to similar circumstances in other disciplines which deal with quantities with dimensions.

**Keywords:** Dimensional Analysis, II-Theorems.

## 1. Introduction

Dimensional analysis is a very simple but powerful toolbox to infer or to verify physical formula. Plenty of examples of its applications can be found in books [1, 2]. The aim of this paper is to investigate another aspect of dimensional analysis, which focuses on its relations to various functions. As for the author he has not encountered treatises or papers that touch upon the topic explicitly, although it is also helpful in order to infer or to verify physical formula. Actually it will be of benefit to know, for instance, that the arguments of exponential and trigonometric functions must be dimensionless.

We deal with phenomena involving some quantities with dimensions and we assume that a system of dimensional units is given throughout. If we are studying physical phenomena, we will need the SI system or the MKS system, for example. In this paper we denote the dimension of a quantity (variable)  $\phi$  by  $\mathbb{D}(\phi)$  instead of the conventional notation  $[\phi]$ , since we are concerned only with relations between dimensions of quantities in general rather than specific physical laws.

Let  $\{A, B, \dots, C\}$  be a minimal system of dimensional units for describing physical phenomena with which we are concerned. This means that any dimensional quantity  $\phi$  has a dimension represented by  $\mathbb{D}(\phi) = A^a B^b \dots C^c$ , where  $a, b, \dots, c$  are all rational numbers, which are uniquely determined by the dimension of  $\phi$ . This assumption is wide enough for dimensional analysis to be applied elsewhere. The following six properties are trivial in physics and we take it for granted that they are valid not only in physics but in other disciplines that deal with dimensional quantities. These provide us with our starting point of the discussions and will be freely utilized.

- (i) For two quantities  $\phi$  and  $\psi$ , the addition  $\phi + \psi$  or the subtraction  $\phi - \psi$  can be defined adequately, if and only if  $\mathbb{D}(\phi) = \mathbb{D}(\psi)$  holds, i.e.,  $\phi$  and  $\psi$  have the same dimension.
- (ii) If  $\mathbb{D}(\phi) = A^a B^b \dots C^c$  and  $k$  is a rational number, then  $\mathbb{D}(\phi^k) = A^{ka} B^{kb} \dots C^{kc}$ .
- (iii) If  $\mathbb{D}(\phi) = A^a B^b \dots C^c$  and  $\mathbb{D}(\psi) = A^d B^e \dots C^f$ , then  $\mathbb{D}(\phi\psi) = A^{a+d} B^{b+e} \dots C^{c+f}$ .

- (iv) For a dimensionless quantity  $\phi$ , we have  $\mathbb{D}(\phi) = A^0 B^0 \cdots C^0$ . We will denote it simply by  $\mathbb{D}(\phi) = 1$ . Otherwise we call  $\phi$  dimensional or a quantity having a dimension, sometimes denoted by  $\mathbb{D}(\phi) \neq 1$ .
- (v) For a natural number  $n$ , if the  $n$ th derivative  $d^n \psi / d\phi^n$  is defined, then  $\mathbb{D}(d^n \psi / d\phi^n) = \mathbb{D}(\psi \phi^{-n})$ .
- (vi) If  $\psi$  is a function of  $\phi$ , then  $\mathbb{D}(\int \psi d\phi) = \mathbb{D}(\psi \phi)$ .

When we regard  $A^a B^b \cdots C^c$  as an algebraic product, (ii) and (iii) can be rewritten as the expressions  $\mathbb{D}(\phi^k) = (\mathbb{D}(\phi))^k$  and  $\mathbb{D}(\phi\psi) = \mathbb{D}(\phi)\mathbb{D}(\psi)$ , respectively.

Some constants may have their own dimensions; Planck's constant  $h$  and Boltzmann's constant  $k$ , for example, are not dimensionless. It should be remarked that constants play an important role in dimensional analysis, too.

Another usage of constants is illustrated by the following example. Let us consider a second-order differential equation  $\psi'' + \psi = 0$  in view of dimensional analysis. Here  $\psi$  is an unknown function of  $\phi$ . To this end we need set  $\psi'' + a\psi = 0$  by introducing a constant  $a$ . Then, putting  $\mathbb{D}(\phi) = A$  and  $\mathbb{D}(\psi) = B$ , we postulate  $a = 1$  and  $\mathbb{D}(a) = A^{-2}$ . We keep  $a$  as such a constant in the course of solving the equation and finally put  $a = 1$ .

The next section is devoted to the investigations of the dimensions for functions that yield physical quantities and the dimensions for the arguments of the functions. In particular, giving simple examples, we show that there are three types of functions from a viewpoint of dimensional analysis. As an illustrative example of application we give simple second-order ordinary differential equations. We refer to an easily available reference [3] that deals with solutions of first-order ordinary differential equations by dimensional analysis. Our results may be applied to some examples of differential equations treated therein. The final section supplies some related remarks.

It is easily recognized that the theory proposed in this paper can also be applied to similar circumstances in other disciplines, e.g., engineering, economics or biology, which involve quantities with dimensions.

## 2. Dimensional Analysis and Functions

In order to describe physical laws we need fundamental functions; exponential, logarithmic and trigonometric functions *etc.* They are building blocks for more complicated functions. We look at these functions from a viewpoint of dimensional analysis.

We can classify such functions  $\psi = \psi(\phi)$  into the following three types in view of dimensions of both  $\phi$  and  $\psi$ . Note that the first two types of functions can take dimensional quantities as their arguments. The examples taken here are the simplest ones exhibiting respective features. Even these simple examples have interesting implications. In the next section a procedure for obtaining functions of respective types will be given that is based on the consideration of this section.

**Type I.** Both  $\phi$  and  $\psi$  can have dimensions.

A typical and obvious example is the power function  $\psi = \phi^r$  with  $r$  a non-zero rational number;  $\phi^3$ ,  $\phi^{-1}$  and  $\sqrt{\phi}$  *etc.* It is obvious that it has a dimension if the argument  $\phi$  has a dimension;  $\mathbb{D}(\psi) = (\mathbb{D}(\phi))^r \neq 1$ .

**Type II.**  $\psi$  is dimensionless, whatever the dimension of  $\phi$  is.

An important example is the logarithmic function. As for the definition of (natural) logarithmic function we take

$$\psi = \ln \phi = \int_1^\phi \frac{d\phi}{\phi}.$$

Therefore, we see that  $\psi$  is dimensionless, since so is the part  $\phi^{-1}d\phi$ , whatever the dimension of  $\phi$  is. Hence we always have

$$\mathbb{D}(\ln \phi) = 1, \quad (1)$$

whether  $\phi$  has a dimension or not.

There are many laws involving the logarithmic function  $\ln \phi$  with  $\mathbb{D}(\phi) \neq 1$  in physics, where  $\phi$  may have a dimension of volume or temperature *etc.* Using (1), it is easy to see that functions of the form  $\psi = \phi^r \ln \phi$  with  $r$  non-zero rational numbers are of Type I.

**Type III.** Both  $\phi$  and  $\psi$  must be dimensionless;  $\mathbb{D}(\phi) = \mathbb{D}(\psi) = 1$ .

Examples for this type include all of trigonometric functions. We show this by  $\psi = \sin \phi$ . We begin with the integral

$$\phi = \int_0^\psi \frac{d\psi}{\sqrt{a^2 - \psi^2}} = \int_0^\psi \frac{d(\psi/a)}{\sqrt{1 - (\psi/a)^2}} = \int_0^{\psi/a} \frac{dx}{\sqrt{1 - x^2}} = \arcsin \frac{\psi}{a} \quad (2)$$

for a positive constant  $a$ . In order for  $a^2 - \psi^2$  to be meaningful,  $\mathbb{D}(\psi) = \mathbb{D}(a)$  must hold and we get  $\mathbb{D}(\psi/a) = 1$ , from which we see that  $\phi$  must be dimensionless. Writing it as  $\sin \phi = \psi/a$ , it also follows that  $\mathbb{D}(\sin \phi) = 1$ . It is easy to see that all other trigonometric functions are of this type.

Another important example of Type III is the exponential function. We have, for some positive constant  $a$ ,

$$\phi = \ln \psi - \ln a = \int_a^\psi \frac{d\psi}{\psi}.$$

Here  $\phi$  is dimensionless, as is shown in Type II, and  $\mathbb{D}(\psi) = \mathbb{D}(a)$ . Rewriting this as  $\psi/a = e^\phi$ , we see that  $\mathbb{D}(e^\phi) = 1$ .

From the discussion of Type III we can observe that whenever the exponential function  $\exp(\cdot)$  appears in physical laws, the argument is always dimensionless. In statistical mechanics a term  $\exp(-E/kT)$  plays a central role, where both  $E$  and  $kT$  have a dimension of energy. So the argument  $-E/kT$  is dimensionless. Similarly, wherever  $\sin(\cdot)$  and  $\cos(\cdot)$  *etc.* appear in physical laws, their arguments must also be dimensionless. Another illustrative example is given by the following.

**Example.** Let a quantity  $\psi$  be a function of a quantity  $\phi$  and let it be governed by a law

$$\psi'' - a^2\psi = 0,$$

where  $a$  is a non-zero constant. It is sufficient to introduce two dimensional units  $\{A, B\}$ ;  $\mathbb{D}(\phi) = A$  and  $\mathbb{D}(\psi) = B$ . Then we get  $\mathbb{D}(a) = A^{-1}$  and  $a\phi$  is the simplest dimensionless

quantity. The solution is of the form  $\psi(\phi) = \alpha \exp(a\phi) + \beta \exp(-a\phi)$  for some constants  $\alpha$  and  $\beta$ , where  $\mathbb{D}(\alpha) = \mathbb{D}(\beta) = B$ .

On the contrary the solution of the equation

$$\psi'' + a^2\psi = 0$$

is of the form  $\psi(\phi) = \alpha \sin a\phi + \beta \cos a\phi$ , where  $\alpha$  and  $\beta$  are the same as above.

### 3. Some Remarks

In this section a procedure for obtaining functions of Type II or III is considered by means of the results in the previous section, and finally a remark will be made on the relation to the  $\Pi$ -theorem in dimensional analysis. It is not a general procedure for the derivation of these functions.

Recalling the fact that the logarithmic function  $\ln(\cdot)$  can take a dimensional quantity as the argument, we are able to derive some functions of Type II. Let us consider the function, for instance,

$$\psi = \int_1^\phi \frac{\sin(\ln \phi) d\phi}{\phi}.$$

By (1) and the fact that  $\sin(\cdot)$  is dimensionless, we see that  $\sin(\ln \phi)$  becomes dimensionless. Hence, using a similar reasoning employed in Type II, we get  $\mathbb{D}(\psi) = 1$  for this function  $\psi$ , whether  $\phi$  has a dimension or not.

In order to present other functions of Type III, let us define the following, for a natural number  $n$ ,

$$y = \int_0^x \frac{dx}{\sqrt[n]{1-x^n}} = G_n(x)$$

and its inverse function  $y = F_n(x)$ . Considering the corresponding equalities in (2) for this case, we get the formula

$$\phi = \int_0^\psi \frac{d\psi}{\sqrt[n]{a^n - \psi^n}} = G_n\left(\frac{\psi}{a}\right).$$

In a similar manner utilized for trigonometric functions (Type III) in section 2 we can obtain  $\mathbb{D}(a) = \mathbb{D}(\psi)$  and  $\mathbb{D}(\sqrt[n]{a^n - \psi^n}) = \mathbb{D}(\psi)$ . Hence we get  $\mathbb{D}(\phi) = 1$ . Thus, from  $\psi/a = F_n(\phi)$ , we can obtain functions  $F_n$  of Type III;  $\mathbb{D}(\phi) = \mathbb{D}(F_n(\phi)) = 1$ . They are generalizations of trigonometric functions. Indeed, when the mathematician C. F. Gauss (1777-1855) attempted to generalize trigonometric functions, he at first introduced and considered these functions  $G_n$  and  $F_n$  prior to the discovery of elliptic functions.

When conventionally dealing with dimensional analysis, we often utilize the  $\Pi$ -theorem ([1, 2]), which requires our efforts to find a function

$$\psi = \Pi(\phi_1, \phi_2, \dots, \phi_m).$$

Here all quantities together with  $\psi$  are dimensionless;  $\mathbb{D}(\phi_1) = \dots = \mathbb{D}(\phi_m) = \mathbb{D}(\psi) = 1$ . Although it is a function of several variables, it seems that our results may be helpful to determine its form.

## References

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